Problem 12-27

When a particle falls through the air, its initial acceleration $a = g$ diminishes until it is zero, and thereafter it falls at a constant or terminal velocity v_f. If this variation of the acceleration can be expressed as $a = (g/v_f^2)(v_f^2 - v^2)$, determine the time needed for the velocity to become $v = v_f/2$. Initially the particle falls from rest.

Solution

The acceleration and velocity are related by

$$a = \frac{dv}{dt} = \frac{g}{v_f^2}(v_f^2 - v^2).$$

Separate variables.

$$\frac{dv}{v^2 - v_f^2} = -\frac{g}{v_f^2}\,dt$$

Integrate both sides.

$$\int_0^{v_f/2} \frac{dv}{v^2 - v_f^2} = \int_0^T -\frac{g}{v_f^2}\,dt$$

$$\int_0^{v_f/2} \frac{dv}{(v + v_f)(v - v_f)} = -\frac{g}{v_f^2}(T - 0)$$

$$\int_0^{v_f/2} \left(\frac{\frac{1}{2v_f}}{v - v_f} - \frac{\frac{1}{2v_f}}{v + v_f} \right) dv = -\frac{g}{v_f^2}T$$

$$\frac{1}{2v_f}\ln|v - v_f|\Big|_0^{v_f/2} - \frac{1}{2v_f}\ln|v + v_f|\Big|_0^{v_f/2} = -\frac{g}{v_f^2}T$$

$$\frac{1}{2v_f}\ln\frac{1}{2} - \frac{1}{2v_f}\ln\frac{3}{2} = -\frac{g}{v_f^2}T$$

$$\frac{1}{2v_f}\ln\frac{1}{3} = -\frac{g}{v_f^2}T$$

$$-\frac{1}{2v_f}\ln 3 = -\frac{g}{v_f^2}T$$

Therefore, the time for the velocity to become $v = v_f/2$ is

$$T = \frac{v_f}{2g}\ln 3.$$